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Non-stationary thermodynamics and wave propagation in heat conducting viscous fluids

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Abstract. The behaviour of heat conducting viscous fluids is described through a suitable set of hidden variables whose (objective) evolution equations account also for cross-effect coupling terms. Such equations are incorporated into a thermodynamic theory which includes Müller's one as a particular case and leads to Navier–Stokes' and Fourier's laws when uniform constant gradients of velocity and temperature are concerned. Meanwhile, the whole nonlinear theory turns out to be hyperbolic; this is shown via a direct analysis of the propagation modes. Finally, we outline an operative way of testing whether the co-rotational derivative is the required objective time derivative.

1. Introduction

Current approaches to non-stationary irreversible thermodynamics are developed on the basis of kinetic theory arguments or hinge on phenomenological standpoints. Within its range of validity the kinetic theory provides more detailed schemes; yet phenomenological theories are needed to corroborate the results by showing that they are not a consequence of a particular model or approximation. Among the phenomenological formulations of non-stationary irreversible thermodynamics, a prominent one is that of Müller (1967). Basically, Müller's paper describes non-equilibrium states by allowing the entropy to depend also on the heat flux and the viscous stress, and, moreover, by accounting for an entropy extra-flux besides the usual term proportional to the heat flux. The transport equations so obtained by Müller, which are substantiated also by a kinetic theory performed previously by Grad (1949), have been subsequently re-examined in different contexts by Lebon and Lambermont (1976) and Kranyš (1977a). General relativistic counterparts of Müller's approach have been accomplished by Israel (1976) and Kranyš (1977b) in the case of fluid mixtures and dissipative elastic media, respectively.

An alternative phenomenological approach to irreversible thermodynamics may be carried out through the model of materials with hidden variables. From a physical viewpoint the hidden variables closely resemble the well known relaxed fluxes of non-stationary thermodynamics (Maugin 1974, Kranyš 1977a). The mathematical aspects pertaining to the hidden variable model are exhibited in the pioneering article by Coleman and Gurtin (1967) and investigated within a different scheme by Day (1976). Recently, appropriate improvements of the model (Morro 1980a) have resulted in a description of viscosity which is hyperbolic in character (Bampi and Morro 1980), besides being compatible with thermodynamics. Motivated by the encouraging results

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achieved so far, in this paper we look again at the hidden variable approach in conjunction with heat conducting viscous fluids.

The primary purpose of this work is to set up a thermodynamic theory of fluids with hidden variables including cross-effect coupling terms in the evolution (or transport) equations. Accordingly, in § 2, after a general outline of the subject, we write the evolution equations involving four coupling terms and we show that, in this case too, the hidden variables do not depend on the present values of the real variables. In view of this prominent property, in § 3 we are able to cast the scheme so achieved within a thermodynamic framework. Owing to the presence of cross terms, we admit the existence of an entropy extra-flux, thus generalising the customary Clausius–Duhem inequality. Then the great flexibility of the hidden variable approach enables us to obtain just Müller's theory through a particular choice of the free energy function.

A second purpose of this paper is to show whether, and how, the new theory accounts for wavefront propagation (§ 4) without any recourse to linear approximations. The hyperbolicity of the full theory is proved simply by exhibiting the propagation modes and the corresponding characteristic speeds of the acceleration waves propagating into a state at equilibrium. The physical interest of these results lies in the possibility of determining the phenomenological coefficients by means of suitable measures of the propagation speeds and the amplitudes of the waves.

A third, minor, aim of this article is to give further insights into the question of the time derivative appearing in the evolution equations. In § 5 we outline how a measure of the transverse wave speed allows us to check whether the co-rotational derivative is suitable as the objective time derivative in the evolution equations.

2. Coupled evolution equations for hidden variables

Throughout, ρ stands for the mass density, θ the absolute temperature, ψ the free energy, η the entropy, e the internal energy, $\boldsymbol{T}$ the stress tensor, $\boldsymbol{q}$ the heat flux, $\boldsymbol{f}$ the body force, and r the heat supply; $\boldsymbol{V}$ is the velocity field relative to a fixed inertial frame $\mathscr{F}$. The symbol $\boldsymbol{\nabla}$ denotes the spatial gradient while $\boldsymbol{D}$ is the symmetric part of the velocity gradient $\boldsymbol{L} = \boldsymbol{\nabla} \boldsymbol{V}$ and a colon indicates complete contraction.

For the purpose of describing non-equilibrium phenomena, here the behaviour of a fluid is expressed through a set of C^2 response functions

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\theta, \rho, \boldsymbol{\alpha}), \tag{1}$$

where $\boldsymbol{\sigma}$ is the array $(\psi, \eta, \boldsymbol{T}, \boldsymbol{q})$, and through differential equations of the form

$$\mathring{\boldsymbol{\alpha}} = \boldsymbol{h}(\theta, \rho, \boldsymbol{D}, \boldsymbol{\nabla}\theta, \boldsymbol{\alpha}, \boldsymbol{\nabla}\boldsymbol{\alpha}) \tag{2}$$

governing the evolution of the hidden variables $\boldsymbol{\alpha}$. The superposed spot in (2) represents an appropriate time derivative. Often such a derivative is identified with the ordinary time derivative, namely, for any vector $\boldsymbol{u}$, $\mathring{\boldsymbol{u}} = \dot{\boldsymbol{u}} \equiv \mathrm{d}\boldsymbol{u}/\mathrm{d}t$. Sometimes, instead, in order to make the constitutive theory frame-indifferent (objective), the spot derivative is taken to be the co-rotational time derivative, that is to say $\mathring{\boldsymbol{u}} = \dot{\boldsymbol{u}} - \boldsymbol{W}\boldsymbol{u}$, with $\boldsymbol{W} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^{\mathrm{T}})$ (Truesdell and Toupin 1960, § 148). Of course the two derivatives are assumed to coincide when acting on scalars. Although the choice of the derivative is not crucial to the results exhibited in §§ 2, 3 and 4, henceforth, for the sake of definiteness, we confine our attention to the co-rotational derivative.

As to the response functions (1) and the evolution equations (2), observe that they do not satisfy the rule of equipresence (Truesdell and Toupin 1960, § 293, Eringen 1975b); the difference between the arguments of (1) and (2) is motivated by the requirement of compatibility with wave propagation (Bampi and Morro 1979, Morro 1980b).

It is possible to present the next developments within the general scheme summarised by equations (1) and (2), through the adoption of very weak restrictions on the function $\boldsymbol{h}$†. However, both to avoid inessential formal difficulties and to obtain a theory providing the most direct generalisation of Navier–Stokes' and Fourier's laws, it is convenient to imagine the hidden variables $\boldsymbol{\alpha}$ as the triple of the symmetric traceless second-order tensor $\boldsymbol{\alpha}_1$, the scalar α_2, and the vector $\boldsymbol{\alpha}_3$, and to choose $\boldsymbol{h}$ so as to make equations (2) into the coupled system

$$\begin{aligned} \mathring{\boldsymbol{\alpha}}_1 &= (1/\tau_1)(\langle \boldsymbol{D}\rangle - \boldsymbol{\alpha}_1) + a\langle \boldsymbol{\nabla}\boldsymbol{\alpha}_3\rangle, \\ \mathring{\alpha}_2 &= (1/\tau_2)(\operatorname{Tr} \boldsymbol{D} - \alpha_2) + b\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}_3, \\ \mathring{\boldsymbol{\alpha}}_3 &= (1/\tau_3)(\boldsymbol{\nabla}\theta - \boldsymbol{\alpha}_3) + c\boldsymbol{\nabla}\alpha_2 + d\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}_1, \end{aligned} \tag{3}$$

where $\langle\ \rangle$ denotes the symmetric traceless part while a, b, c, d are phenomenological coefficients and τ_1, τ_2, τ_3 are relaxation times.

Owing to the structure of the evolution equations commonly used in the literature, we have become accustomed to the profitable feature of the hidden variables $\boldsymbol{\alpha}$ whereby the values $\boldsymbol{\alpha}(t)$ are independent of the values, at the same time t, of the real variables. To show that this feature occurs here as well, consider, for example, the equation $(3)_3$; at any particle of the fluid a trivial integration yields

$$\begin{aligned} \boldsymbol{\alpha}_3(t) = \int_{t_0}^{t} & \exp[-(t-\xi)/\tau_3](\boldsymbol{\nabla}\theta/\tau_3 + c\boldsymbol{\nabla}\alpha_2 + d\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}_1 + \boldsymbol{W}\boldsymbol{\alpha}_3)(\xi)\,\mathrm{d}\xi \\ & + \boldsymbol{\alpha}_3(t_0)\exp[-(t-t_0)/\tau_3]. \end{aligned} \tag{4}$$

Now, given a set of C^1 functions (histories) θ', ρ', $\boldsymbol{D}'$, $(\boldsymbol{\nabla}\theta)'$, look at C^1 functions θ'', ρ'', $\boldsymbol{D}''$, $(\boldsymbol{\nabla}\theta)''$ such that $\theta' = \theta''$, $\rho' = \rho''$, $\boldsymbol{D}' = \boldsymbol{D}''$, $(\boldsymbol{\nabla}\theta)' = (\boldsymbol{\nabla}\theta)''$ in $[t_0, t-\epsilon]$ and that $\theta''(t)$, $\rho''(t)$, $\boldsymbol{D}''(t)$, $(\boldsymbol{\nabla}\theta)''(t)$, besides $\dot{\theta}''(t)$, are arbitrary. Then by virtue of (4), it is evident that the choice of a small enough ϵ makes as small as we please the change of $\boldsymbol{\alpha}_3(t)$ induced by the replacement of θ', ρ', $\boldsymbol{D}'$, $(\boldsymbol{\nabla}\theta)'$ with θ'', ρ'', $\boldsymbol{D}''$, $(\boldsymbol{\nabla}\theta)''$, independently of the values taken by $\boldsymbol{\nabla}\alpha_2$, $\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}_1$, and $\boldsymbol{W}$. Analogous proofs hold also for $\boldsymbol{\alpha}_1$ and α_2.

3. Hidden variable approach to non-stationary thermodynamics

Having established the evolution properties, we are able to incorporate the hidden variables $\boldsymbol{\alpha}_1$, α_2, $\boldsymbol{\alpha}_3$ into a thermodynamic framework, once the appropriate form of the second law is decided. As is well known, conventional theories postulate that entropy flux is simply proportional to heat flux (see e.g. Coleman 1964). More general theories (see e.g. Eringen 1975a, § 3.3) allow for the existence of an entropy extra-flux $\boldsymbol{N}$ in addition to the usual term $\boldsymbol{q}/\theta$. In isotropic fluids the simplest form of $\boldsymbol{N}$ involving the hidden variables $\boldsymbol{\alpha}_1$, α_2, and $\boldsymbol{\alpha}_3$ is

$$\boldsymbol{N} = K\boldsymbol{\alpha}_1\boldsymbol{\alpha}_3 + L\alpha_2\boldsymbol{\alpha}_3; \tag{5}$$

† Here we have in mind the restrictions considered by Day (1976).

a possible connection between the phenomenological coefficients K, L and a, b, c, d will be given shortly. Accordingly, on account of the usual balance equations

$$\dot{\rho}+\rho\nabla\cdot\boldsymbol{V}=0, \qquad \rho\dot{\boldsymbol{V}}-\nabla\cdot\boldsymbol{T}=\rho f, \qquad \rho\dot{e}-\boldsymbol{T}:\boldsymbol{D}+\nabla\cdot\boldsymbol{q}=\rho r, \quad (6)$$

the second law of thermodynamics may be given the form of the inequality

$$-\rho(\dot{\psi}+\eta\dot{\theta})+\boldsymbol{T}:\boldsymbol{D}-(1/\theta)\boldsymbol{q}\cdot\nabla\theta+\theta\nabla\cdot\boldsymbol{N}\geqslant 0 \quad (7)$$

which must hold for any particle. Then substitution of equations (3) into (7) provides

$$\begin{aligned}-\rho(\psi_\theta+\eta)\dot{\theta}+\Big(\boldsymbol{T}-\frac{\rho}{\tau_1}\psi_{\boldsymbol{\alpha}_1}\Big):\langle\boldsymbol{D}\rangle+\Big(\rho^2\psi_\rho+\tfrac{1}{3}\operatorname{Tr}\boldsymbol{T}-\frac{\rho}{\tau_2}\psi_{\alpha_2}\Big)\operatorname{Tr}\boldsymbol{D}\\ -\Big(\frac{1}{\theta}\boldsymbol{q}+\frac{\rho}{\tau_3}\psi_{\boldsymbol{\alpha}_3}\Big)\cdot\nabla\theta+\rho\Big(\frac{1}{\tau_1}\psi_{\boldsymbol{\alpha}_1}:\boldsymbol{\alpha}_1+\frac{1}{\tau_2}\psi_{\alpha_2}\alpha_2+\frac{1}{\tau_3}\psi_{\boldsymbol{\alpha}_3}\cdot\boldsymbol{\alpha}_3\Big)\\ +(\theta K\boldsymbol{\alpha}_1-\rho a\psi_{\boldsymbol{\alpha}_1}):\nabla\boldsymbol{\alpha}_3+(\theta L\alpha_2-\rho b\psi_{\alpha_2})\nabla\cdot\boldsymbol{\alpha}_3\\ +(\theta L\boldsymbol{\alpha}_3-\rho c\psi_{\boldsymbol{\alpha}_3})\cdot\nabla\alpha_2+(\theta K\boldsymbol{\alpha}_3-\rho\, d\psi_{\boldsymbol{\alpha}_3})\cdot(\nabla\cdot\boldsymbol{\alpha}_1)\geqq 0,\end{aligned} \quad (8)$$

where the subscripts θ, ρ, $\boldsymbol{\alpha}_1$, α_2, $\boldsymbol{\alpha}_3$ denote partial differentiations. Now, on appealing to the independence of the hidden variables $\boldsymbol{\alpha}_1(t)$, $\alpha_2(t)$, $\boldsymbol{\alpha}_3(t)$, and hence of $\psi(t)$, of the present values $\dot{\theta}(t)$, $\boldsymbol{D}(t)$, $(\nabla\theta)(t)$, we conclude that the inequality (8) holds identically if and only if

$$\begin{aligned}\eta&=-\psi_\theta,\\ \boldsymbol{T}&=-p\boldsymbol{I}+(\rho/\tau_1)\psi_{\boldsymbol{\alpha}_1}+(\rho/\tau_2)\psi_{\alpha_2}\boldsymbol{I},\\ \boldsymbol{q}&=-(\rho\theta/\tau_3)\psi_{\boldsymbol{\alpha}_3},\end{aligned} \quad (9)$$

$p=\rho^2\psi_\rho$ being the pressure, and

$$\begin{aligned}\rho\Big(\frac{1}{\tau_1}\psi_{\boldsymbol{\alpha}_1}:\boldsymbol{\alpha}_1+\frac{1}{\tau_2}\psi_{\alpha_2}\alpha_2+\frac{1}{\tau_3}\psi_{\boldsymbol{\alpha}_3}\cdot\boldsymbol{\alpha}_3\Big)+(\theta K\boldsymbol{\alpha}_1-\rho a\psi_{\boldsymbol{\alpha}_1}):\nabla\boldsymbol{\alpha}_3+(\theta L\alpha_2-\rho b\psi_{\alpha_2})\nabla\cdot\boldsymbol{\alpha}_3\\ +(\theta L\boldsymbol{\alpha}_3-\rho c\psi_{\boldsymbol{\alpha}_3})\cdot\nabla\alpha_2+(\theta K\boldsymbol{\alpha}_3-\rho\, d\psi_{\boldsymbol{\alpha}_3})\cdot(\nabla\cdot\boldsymbol{\alpha}_1)\geqslant 0.\end{aligned} \quad (10)$$

Whenever the function ψ satisfies the inequality (10), the response functions η, $\boldsymbol{T}$, $\boldsymbol{q}$ defined by (9) are automatically consistent with the second law of thermodynamics in the form (7).

From a merely mathematical standpoint, we cannot prefer any function ψ among all those compatible with (10). Physical arguments, instead, allow us to select the particular free energy function ψ given by

$$\psi=\Psi(\theta,\rho)+\frac{1}{\rho}\Big(\mu\tau_1\boldsymbol{\alpha}_1:\boldsymbol{\alpha}_1+\tfrac{1}{2}\zeta\tau_2\alpha_2^2+\frac{\kappa\tau_3}{2\theta}\boldsymbol{\alpha}_3\cdot\boldsymbol{\alpha}_3\Big). \quad (11)$$

First, owing to the independence of the quantities $\nabla\cdot\boldsymbol{\alpha}_1$, $\nabla\alpha_2$, $\nabla\boldsymbol{\alpha}_3$, and $\nabla\cdot\boldsymbol{\alpha}_3$, equation (11) and the identical validity of the inequality (10) imply that

$$2\mu\tau_1 a=\theta K, \qquad \zeta\tau_2 b=\theta L, \qquad \kappa\tau_3 c=\theta^2 L, \qquad \kappa\tau_3 d=\theta^2 K, \quad (12)$$

and that $\mu\geqslant 0$, $\zeta\geqslant 0$, $\kappa\geqslant 0$. The conditions (12) reduce the effective number of the phenomenological coefficients to two only. Relations of this sort are not new in the literature (see e.g. Müller 1967, Kranyš 1977a) and are sometimes referred to as a generalisation of the Onsager reciprocal relations (see e.g. Stewart 1977, p 66); here they are a genuine consequence of the second law of thermodynamics. Second, equations (9) and (11) give

$$\eta = -\Psi_\theta + (\kappa\tau_3/2\rho\theta^2)\boldsymbol{\alpha}_3 \cdot \boldsymbol{\alpha}_3 \tag{13}$$

and the constitutive equations

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu\boldsymbol{\alpha}_1 + \zeta\alpha_2\boldsymbol{I}, \qquad \boldsymbol{q} = -\kappa\boldsymbol{\alpha}_3, \tag{14}$$

which are the most natural generalisation of Navier–Stokes' and Fourier's laws. To make this assertion clear, look at time-independent uniform fields $\boldsymbol{D}$, $\nabla\theta$ and assume that $\boldsymbol{W} = \boldsymbol{0}$; asymptotically the equations (3) yield

$$\boldsymbol{\alpha}_1 = \langle\boldsymbol{D}\rangle, \qquad \alpha_2 = \mathrm{Tr}\,\boldsymbol{D}, \qquad \boldsymbol{\alpha}_3 = \nabla\theta, \tag{15}$$

whereby the hidden variables coincide with the corresponding real variables, and what is more the equations (14) become the standard laws of viscosity and heat conduction.

The results assembled above are easily shown to be in strict connection with Müller's (1967). Indeed, on expressing the hidden variables $\boldsymbol{\alpha}_1, \alpha_2, \boldsymbol{\alpha}_3$ in terms of $\langle\boldsymbol{T}\rangle$, Tr $\boldsymbol{T}$, $\boldsymbol{q}$ via equations (14) and substituting into (3), we obtain Müller's transport equations provided that simple identifications among the corresponding coefficients are made.

4. Acceleration waves propagating into a region at equilibrium

Henceforth we consider the propagation of acceleration wavefronts (acoustic waves) in fluids described by the constitutive equations (11), (13) and (14). In this connection we note that the independence of the hidden variables $\boldsymbol{\alpha}_1(t)$, $\alpha_2(t)$, $\boldsymbol{\alpha}_3(t)$ of the present values $\boldsymbol{D}(t)$, $\nabla\theta(t)$ enables us to assume the continuity of $\boldsymbol{\alpha}_1$, α_2, $\boldsymbol{\alpha}_3$ even though $\boldsymbol{D}$, $\nabla\theta$ suffer jump discontinuities as happens at acceleration wavefronts. Accordingly, we say that a moving singular surface $s(t)$ represents an acceleration wave if the functions $\boldsymbol{V}$, θ, ρ, $\boldsymbol{\alpha}_1$, α_2, $\boldsymbol{\alpha}_3$ are continuous everywhere, whereas their time and spatial derivatives of any order suffer jump discontinuities across $s(t)$ but are continuous functions everywhere else. In order to avoid overly cumbersome expressions of the propagation modes and the related speeds, in this section the fluid ahead of the front is assumed to be held at equilibrium—$\boldsymbol{L} = \boldsymbol{0}$, $\nabla\theta = \boldsymbol{0}$—until the arrival of the wave at time t. In such a case, no matter how the initial values for the hidden variables at $t_0 = -\infty$ are chosen, we may take $\boldsymbol{\alpha}_1(t) = \boldsymbol{0}$, $\alpha_2(t) = 0$, $\boldsymbol{\alpha}_3(t) = \boldsymbol{0}$. To prove that this is so, we begin by remarking that if $\boldsymbol{L} = \boldsymbol{0}$, $\nabla\theta = \boldsymbol{0}$ at all times prior to t, the evolution equations (3) allow us to find that α_2 can be written as

$$\alpha_2(t) = \exp[-(t - t_0)(\tau_2 + \tau_3)/\tau_2\tau_3]y,$$

where, asymptotically, y is the solution of the Klein–Gordon equation

$$\ddot{y} - bc\nabla^2 y - [(\tau_2 - \tau_3)/(2\tau_2\tau_3)]^2 y = 0.$$

Consideration of this result shows that the same is true for $\boldsymbol{\alpha}_3$. Hence we find that $\boldsymbol{\alpha}_1$ too tends monotonically to zero. Accordingly the condition $t_0 = -\infty$ provides the desired conclusion.

To proceed further we need two compatibility relations for the discontinuities across s. Consider any time-dependent field ϕ and denote by [ ] the jump across the front s.

The first relation is supplied by Maxwell's theorem. Let $\boldsymbol{n}$ be the unit normal to s and set $\nabla_n = \boldsymbol{n} \cdot \nabla$; then $[\phi] = 0$ implies that

$$[\nabla\phi] = [\nabla_n\phi]\boldsymbol{n}. \tag{16}$$

To derive the second relation, look at an observer $\mathscr{D}$ moving with the surface s at the speed $\boldsymbol{V} \cdot \boldsymbol{n} + U$, relative to $\mathscr{F}$. As is well known, the time-rate of change of ϕ as apparent to $\mathscr{D}$ is given by the equation (see e.g. McCarthy 1975)†

$$\delta\phi/\delta t = \dot{\phi} + U\nabla_n\phi. \tag{17}$$

Hence we obtain

$$\delta[\phi]/\delta t = [\dot{\phi}] + U[\nabla_n\phi].$$

Thus, in the case when $[\phi] = 0$, we find that

$$[\dot{\phi}] = -U[\nabla_n\phi]. \tag{18}$$

To derive the jump relations across the front, we begin by applying the conditions (16) and (18) to the evolution equations (3); from $(6)_1$ it follows that

$$\begin{aligned}
[\dot{\boldsymbol{\alpha}}_1]\boldsymbol{n} &= \left\{ -\frac{2}{3}\frac{aU}{\tau_3(U^2-\gamma)}[\nabla_n\theta] + \left[\frac{2aU}{3\rho(U^2-\gamma)}\left(\frac{c}{\tau_2}+\frac{2d}{3\tau_1}\right) + \frac{2U}{3\tau_1\rho}\right][\nabla_n\rho]\right\}\boldsymbol{n} \\
&\quad + \tfrac{1}{2}(1/\tau_1 + a\mathscr{T})[\nabla_n\boldsymbol{v}], \\
[\dot{\alpha}_2] &= -\frac{bU}{\tau_3(U^2-\gamma)}[\nabla_n\theta] + \left[\frac{bU}{\rho(U^2-\gamma)}\left(\frac{c}{\tau_2}+\frac{2d}{3\tau_1}\right) + \frac{U}{\rho\tau_2}\right][\nabla_n\rho], \\
[\dot{\boldsymbol{\alpha}}_3] &= \frac{U^2}{U^2-\gamma}\left[\frac{1}{\tau_3}[\nabla_n\theta] - \frac{1}{\rho}\left(\frac{c}{\tau_2}+\frac{2d}{3\tau_1}\right)[\nabla_n\rho]\right]\boldsymbol{n} - U\mathscr{T}[\nabla_n\boldsymbol{v}],
\end{aligned} \tag{19}$$

where $\boldsymbol{v} = \boldsymbol{n} \times (\boldsymbol{V} \times \boldsymbol{n})$ and $\gamma = bc + 2ad/3$, $\mathscr{T} = (2U^2 - ad)^{-1}d/\tau_1$. Then, starting with the balance equations $(6)_{2,3}$, some lengthy but straightforward calculations lead us to the jump relations

$$\begin{aligned}
&\left[U^2 - \left(p_\rho + \frac{4\mu}{3\rho\tau_1} + \frac{\zeta}{\rho\tau_2}\right) - \frac{1}{\rho(U^2-\gamma)}(\tfrac{4}{3}\mu a + \zeta b)\left(\frac{c}{\tau_2}+\frac{2d}{3\tau_1}\right)\right][\nabla_n\rho]\boldsymbol{n} \\
&\qquad + \left(\frac{1}{\tau_3(U^2-\gamma)}(\tfrac{4}{3}\mu a + \zeta b) - p_\theta\right)[\nabla_n\theta]\boldsymbol{n} + \left(\rho U - \frac{\mu}{\tau_1 U} - \frac{a\mu}{U}\mathscr{T}\right)[\nabla_n\boldsymbol{v}] = 0, \\
&\left[\rho\theta\eta_\rho + \frac{\kappa}{\rho(U^2-\gamma)}\left(\frac{c}{\tau_2}+\frac{2d}{3\tau_1}\right)\right][\nabla_n\rho] - \left(\frac{\kappa}{\tau_3(U^2-\gamma)} - \rho\theta\eta_\theta\right)[\nabla_n\theta] = 0.
\end{aligned} \tag{20}$$

One glance at these relations tells us that transverse waves can occur. More precisely, letting $\boldsymbol{w}$ be the ordered array $([\nabla_n\rho], [\nabla_n\theta], [\nabla_n\boldsymbol{v}])$, the propagation mode

$$\boldsymbol{w} = (0, 0, [\nabla_n\boldsymbol{v}])$$

corresponding to hydrodynamic homothermal waves travelling at the speeds

$$U_{\mathrm{T}} = \pm(\mu/\rho\tau_1 + \tfrac{1}{2}ad)^{1/2}. \tag{21}$$

Longitudinal waves occur as well; they may be viewed as the counterpart of the

† It is worth emphasising that equation (17) involves the material time derivative $\dot{\phi}$, not the co-rational time derivative.

customary acoustic waves. However, owing to the contributions of heat conductivity and viscosity, here we have fast and slow longitudinal waves travelling at the speeds

$$U_f^2 = \gamma + \lambda + (\lambda^2 - \omega)^{1/2}, \qquad U_s^2 = \gamma + \lambda - (\lambda^2 - \omega)^{1/2}, \tag{22}$$

respectively, where

$$\lambda = p_\rho - \frac{\eta_\rho}{\eta_\theta} p_\theta + \frac{4\mu}{3\rho\tau_1} + \frac{\zeta}{\rho\tau_2} + \frac{\kappa}{\rho\theta\eta_\theta\tau_3} - \gamma,$$

$$\omega = \frac{4}{\rho\theta\eta_\theta}\left[\frac{\kappa}{\tau_3}\left(p_\rho + \frac{4\mu}{3\rho\tau_1} + \frac{\zeta}{\rho\tau_2}\right) - (\tfrac{4}{3}\mu a + \zeta b)\left(\frac{\rho\theta\eta_\rho}{\tau_3} + \frac{c\theta\eta_\theta}{\tau_2} + \frac{2d\theta\eta_\theta}{\tau_1}\right) + \frac{\kappa p_\theta}{\rho}\left(\frac{c}{\tau_2} + \frac{2d}{3\tau_1}\right) - \frac{\kappa\gamma}{\tau_3}\right].$$

The precise forms of the associated propagation modes

$$w = ([\nabla_n \rho], [\nabla_n \theta], \mathbf{0})$$

can be easily deduced from any of the relations (20).†

It is a hard mathematical task to show the realness and the positiveness of the right-hand sides of (22) in their present form. On the other hand, from a physical standpoint it seems reasonable to regard K, L, and hence a, b, c, d, as small parameters, in that the corresponding cross-effects are to be viewed as terms of higher order in the deviations from the local equilibrium. This suggests the examination of (22) within the linear approximation with respect to K, L. In this instance, the desired conclusion is a consequence of the usual thermodynamic properties $\eta_\theta > 0$, $p_\rho > 0$ which force $\lambda_0 > 0$, $\lambda_0^2 - \omega_0 > 0$, where

$$\lambda_0 = p_\rho - \frac{\eta_\rho}{\eta_\theta} p_\theta + \frac{4\mu}{3\rho\tau_1} + \frac{\zeta}{\rho\tau_2} + \frac{\kappa}{\rho\theta\eta_\theta\tau_3}, \qquad \omega_0 = \frac{4\kappa}{\rho\theta\eta_\theta\tau_3}\left(p_\rho + \frac{4\mu}{3\rho\tau_1} + \frac{\zeta}{\rho\tau_2}\right),$$

are the zero-order approximations to λ and ω, respectively.

As a final remark we point out the operative meaning of the results assembled above. More precisely, measures of the local speeds of propagations U_T, U_f, U_s, and the ratios between $[\nabla_n \rho]$ and $[\nabla_n \theta]$ in the case of the longitudinal waves give five relations in the five unknown phenomenological parameters τ_1, τ_2, τ_3, and K, L. So, at least in principle, this fact provides a precise experimental check of the validity of the present theory.

5. Comments about the time derivative

In view of the assumption that the region ahead of the front is at equilibrium, the results of § 4 rule out the possibility of setting up experimental procedures which distinguish the appropriate spot derivative. Here we point out how a particular situation makes an experimental procedure practicable. More precisely, we look at plane waves propagating in a non-heat conducting fluid whose evolution equations involve the co-rotational time derivative as spot derivative. In this case we have†

$$\dot{\boldsymbol{\alpha}}_1 = (1/\tau_1)(\langle \boldsymbol{D} \rangle - \boldsymbol{\alpha}_1) + \boldsymbol{W}\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_1 \boldsymbol{W}, \qquad \dot{\alpha}_2 = (1/\tau_2)(\mathrm{Tr}\, \boldsymbol{D} - \alpha_2). \tag{23}$$

Suppose now that the tensor $\boldsymbol{L}$ in the unperturbed state of the fluid satisfies the

† Observe that, owing to the structure of (12), the vanishing of any of the coefficients μ, ζ, κ cancels every effect of the corresponding hidden variables on the other ones (e.g. $\kappa = 0$ implies that $a = 0$, $b = 0$).

conditions $\boldsymbol{L}=\boldsymbol{D}$, $\boldsymbol{Le}=0$, $\boldsymbol{e}$ being a fixed unit vector. Then, by virtue of (23) and the continuity of $\boldsymbol{\alpha}_1$, it follows that $\boldsymbol{\alpha}_1\boldsymbol{e}=(\boldsymbol{e}\cdot\boldsymbol{\alpha}_1\boldsymbol{e})\boldsymbol{e}$ at the wave. Following the line of calculation of § 4, it is an easy matter to see that transverse waves may exist if the normal $\boldsymbol{n}$ is orthogonal to $\boldsymbol{e}$ and the polarisation $[\nabla_n \boldsymbol{v}]$ is along the direction $\boldsymbol{e}$ itself. The corresponding local speeds of propagation are given by

$$U^2=(\mu/\rho\tau_1)+(\mu/\rho)(\boldsymbol{n}\cdot\boldsymbol{\alpha}_1\boldsymbol{n}-\boldsymbol{e}\cdot\boldsymbol{\alpha}_1\boldsymbol{e}) \tag{24}$$

where the new contribution $(\boldsymbol{n}\cdot\boldsymbol{\alpha}_1\boldsymbol{n}-\boldsymbol{e}\cdot\boldsymbol{\alpha}_1\boldsymbol{e})\mu/\rho$ is a direct consequence of the term $\boldsymbol{W}\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_1\boldsymbol{W}$ in (23)‡. This provides an experimental way of testing if the choice of the co-rotational derivative, appearing in the present evolution equations, is the proper one.

Besides being interesting on its own in the classical context, the problem of finding a proper objective time derivative deserves further attention also in the general relativistic context. There, two slightly different derivatives are usually adopted (see Carter and Quintana 1972, Maugin 1978). The availability of a precise classical time derivative would constitute a valuable insight into the question of the object relativistic time derivative.

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‡ On account of (23), the occurrence of the difference $\boldsymbol{n}\cdot\boldsymbol{\alpha}_1\boldsymbol{n}-\boldsymbol{e}\cdot\boldsymbol{\alpha}_1\boldsymbol{e}$ tells us that, ultimately, the effective contribution to U^2 is due to the quantity $\boldsymbol{n}\cdot\boldsymbol{Dn}$ only.